2705 [May, 1918]. Proposed by PAUL CAPRON, U. S. Naval Academy.

Find the area of a loop of the trochoid

$$x = \tfrac{1}{3}a(3\phi - \pi \sin \phi), \qquad y = \tfrac{1}{3}a(3 - \pi \cos \phi).$$

SOLUTION BY L. E. MENSENKAMP, Freeport, Illinois.

The curve of the type here given is generated by causing a circle of radius a to roll on a straight line. Then, a point on a fixed radius of this circle, and at a distance $\pi a/3$ from the center, will describe the given trochoid. The axis of X is the line on which the circle rolls, and the axis of Y is taken as the position of the radius containing the generating point when it is pointing vertically downward.

The desired area may be found by evaluating $\int y\,dx$ between the proper limits of integration. It is convenient to express this integral in terms of the parameter ϕ, as

$$\int \frac{a^2}{9}(3 - \pi \cos \phi)^2 d\phi.$$

From the way in which the curve is defined above, it is symmetrical with respect to the Y-axis; therefore, it will suffice to find that portion of the loop to the left of the Y-axis and multiply it by 2. As ϕ starts from zero and increases in value, the point which generates the curve starts from its position on the negative portion of the Y-axis and traverses the left half of the loop in a clock-wise direction about the origin, finally returning to the Y-axis again, this time in the upper half of the coördinate plane. The limits of integration in ϕ must, therefore, correspond to the two points where the loop cuts the Y-axis. Examination of the equations of the curve shows these to be 0 and $\pi/6$.

The area of half the curve is therefore given by

$$\frac{a^2}{9}\int_0^{\pi/6}(9 - 6\pi \cos \phi + \pi^2 \cos^2 \phi)d\phi,$$

or

$$\frac{a^2}{9}\left(9\phi - 6\pi \sin \phi + \frac{\pi^2\phi}{2} + \frac{\pi^2}{4}\sin 2\phi\right)_0^{\pi/6}.$$

The area of the entire loop will then be double this, or

$$\frac{a^2\pi}{108}(-36 + 2\pi^2 + 3\pi\sqrt{3}).$$

Also solved by J. B. REYNOLDS and the PROPOSER.

2707 [May, 1918]. Proposed by S. A. COREY, Albia, Iowa.

Let A, B, and C be the vector sides of a triangle. Construct another triangle with vector sides R, S, and T, where

$$R = mA - dnB, \qquad S = nA + (m + en)B.$$

Then prove that

$$(m^2 + emn + dn^2)(a^2 + eab \cos(AB) + db^2) = r^2 + ers \cos(RS) + ds^2,$$

where d, e, m and n are any scalar quantities; a, b, r and s are the tensors, or lengths, of the sides A, B, R and S, respectively; and $\cos(RS)$ is the cosine of the angle between R and S when placed coinitially.

SOLUTION BY THE PROPOSER.

We have the algebraic identity

$$(m^2 + emn + dn^2)(a^2 + eab + db^2) = r^2 + ers + ds^2, \tag{1}$$

where $r = ma - dnb$, and $s = na + (m + en)b$.

As (1) is a quadratic identity in a, b, r, and s, it holds if these quantities be vectors (or quaternions), provided we consider the scalar part only of the vector products which we would have in (1) under such an interpretation. But such an interpretation gives us exactly the equation of the given problem, which is therefore true.

NOTE.—I have used capitals A, B, C, R, S, and T in some cases where small letters are used in the problem as published in the May MONTHLY. This substitution of capitals for small letters is done for the sake of clearness in distinguishing vectors from their tensors.

2708 [May, 1918]. Proposed by WILLIAM HOOVER, Columbus, Ohio.

A uniform plank of length $2a$ and thickness $2h$ rests in equilibrium on a fixed rough horizontal cylinder of radius c, so that a vertical plane containing the dimension $2a$ and the center of gravity of the plank is at right angles to the axis of the cylinder; find the period of a complete small oscillation of the plank.

SOLUTION BY THE PROPOSER.

Let G be the center of gravity of the plank; G_0 the initial position of G; O the center of gravity of the cylinder and vertically beneath G_0; $\varphi =$ the angular rotation of the plank after any time t from the beginning of motion, OX, OY the horizontal and vertical coördinate axes $OA = x$, $GA = y$, the coördinates of G, $k = \sqrt{(a^2 + h^2)/3} =$ the radius of gyration of the plank about an axis parallel to the axis of the cylinder. We obtain

$$x = (c + h) \sin \varphi - c\varphi \cdot \cos \varphi, \tag{1}$$

$$y = (c + h) \cos \varphi + c\varphi \cdot \sin \varphi. \tag{2}$$

By *vis viva*,

$$\tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + k^2\dot{\varphi}^2) = - mgy + C. \tag{3}$$

From (1),

$$\dot{x} = (h \cos \varphi + c\varphi \sin \varphi)\dot{\varphi}, \tag{4}$$

$$\dot{y} = (- h \sin \varphi + c\varphi \cos \varphi)\dot{\varphi}; \tag{5}$$

(3) then is

$$\frac{m}{2} (h^2 + k^2 + c^2\varphi^2)\dot{\varphi}^2 = C - mg\{(c + h) \cos \varphi + c\varphi \sin \varphi\}. \tag{6}$$

Differentiating both sides of (6) with respect to t using the value of k,

$$3c^2\varphi \cdot \dot{\varphi}^2 + \{(a^2 + 4h^2) + c^2\varphi^2\}\ddot{\varphi} = - 3g\{- h \sin \varphi + c\varphi \cos \varphi\}. \tag{7}$$

Let φ be so small that we may put $\sin \varphi \doteq \varphi$, $\cos \varphi = 1$, and omit φ^2, $\dot{\varphi}^2$, etc., because of the nature of the oscillation; we have, then,

$$(a^2 + 4h^2)\ddot{\varphi} = - 3g(c - h) \cdot \varphi, \tag{8}$$

an harmonic equation in φ if $c > h$, giving the period required,

$$T = 2\pi \sqrt{\frac{a^2 + 4h^2}{3g(c - h)}}. \tag{9}$$

It may be instructive to derive (8) by another method given by Holditch in the eighth volume of the *Cambridge Transactions* and quoted by Routh, *Dynamics of a System of Rigid Bodies, Elementary Part*, fourth edition, 1882, pages 341–342.

Let the motion of a body in space of two dimensions be given by the coördinates x, y of its center of gravity, and the angle φ which any fixed line in the body makes with a line fixed in space; $\alpha =$ the equilibrium value of φ; x', x'', etc., denoting $dx/d\varphi$, etc.; x_0', etc., the values of x', etc., when $\varphi = \alpha$, and $k =$ the principal radius of gyration; then

$$(x_0'^2 + k^2)\ddot{\varphi} = - gy_0''\varphi. \tag{10}$$

From (4) and (5) we have, with $\varphi_0 = 0$,

$$x_0' = h, \qquad y_0'' = c - h \tag{11}$$

(11) in (10) gives (8).

Also solved by R. C. COLWELL and J. B. REYNOLDS.

2711 [June, 1918]. Proposed by PAUL CAPRON, U. S. Naval Academy.

Show that the curves (*a*) $a^8y_1^2 = x^4(a^2 - x^2)^3$, (*b*) $a^8y_2^2 = x^8(a^2 - x^2)$ bound ten areas, of which two are each $(a^2/4)(\frac{1}{4}\pi - \frac{1}{3})$ and the remaining eight are each $a^2/24$.